An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 44 (2002) 909–923
www.elsevier.com/locate/camwa

The Balance Space Approach to Multicriteria Decision Making—Involving the Decision Maker

M. EHRGOTT
Department of Engineering Science, University of Auckland
Private Bag 92019, Auckland, New Zealand
m.ehrgott@auckland.ac.nz

Abstract—The balance space approach (introduced by Galperin in 1990) provides a new view on multicriteria optimization. Looking at deviations from global optimality of the different objectives, balance points and balance numbers are defined when either different or equal deviations for each objective are allowed. Apportioned balance numbers allow the specification of proportions among the deviations. Through this concept, the decision maker can be involved in the decision process. In this paper, we prove that the apportioned balance number can be formulated by a min-max operator. Furthermore, we prove some relations between apportioned balance numbers and the balance set, and see the representation of balance numbers in the balance set. The main results are necessary and sufficient conditions for the balance set to be exhaustive, which means that by multiplying a vector of weights (proportions of deviation) with its corresponding apportioned balance number a balance point is attained. The results are used to formulate an interactive procedure for multicriteria optimization. All results are illustrated by examples.

Keywords—Multicriteria decision making, Interactive methods, Balance space, Pareto optimality, Min-max problem.

1. INTRODUCTION

In [1], Galperin introduced a new approach to multiple criteria optimization problems: the balance space. With the balance space approach, multiobjective optimization problems with conflicting objectives can be solved by global optimization methods, see [1,2]. The balance space approach is based on minimal deviations from optimality of the individual objectives. Considering either equal or different deviations for each objective function, the concept of the balance number, respectively, the balance point are derived. In this paper, we focus on a variant that allows an active role of the decision maker, as he can specify proportions among the deviations from optimality according to his purposes. This concept is called apportioned balance number.

In this introductory section, we briefly discuss the approach, present the essential definitions, and review existing results. In Section 2, we present a min-max formulation for the computation of the apportioned balance number. In Section 3, we prove some relations between apportioned balance numbers and balance points. These show that the former are always represented in the balance set. Main results are contained in Section 4. Here we prove sufficient (and in the bicriteria case, necessary conditions) for the balance set to be exhaustive. This means that an apportioned

0898-1221/02/$ - see front matter Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(02)00203-1

balance number multiplied with its weight vector is a balance point. Finally, based on the results of our research, the outline of an interactive algorithm for multicriteria optimization is given in Section 5. The paper is concluded by some comments on topics for future research (Section 6).

Consider a compact set $X \subset \mathbb{R}^n$ and the multiobjective optimization problem

$$\min_{x \in X} f(x) = (f_i(x), \dots, f_m(x)), \tag{1}$$

where we assume all components f_i of f to be continuous. For each i, the corresponding single objective subproblem of (1) has a global optimal solution over the compact set X represented by the partial global minimum value

$$c_i^0 := \min_{x \in X} f_i(x) \tag{2}$$

and the corresponding set of all global minimizers

$$X_i^0 := \left\{ x \in X : f_i(x) = c_i^0 \right\}. \tag{3}$$

If there is a nonempty intersection

$$X^0 := \bigcap_{i=1}^{m} X_i^0 \neq \emptyset, \tag{4}$$

then the multiobjective optimization problem (1) is called balanced, otherwise unbalanced [1, Chapter 8]. However, the presence of conflicting objectives means that the problem is unbalanced in general. Throughout the paper, we will only consider unbalanced problems. Then we can relax the minimization requirements and look for the uniform η-suboptimal solutions

$$X_i^0(\eta) := \left\{ x \in X : f_i(x) - c_i^0 \leq \eta,\ \eta > 0 \right\}. \tag{5}$$

With increasing η, the intersection of $X_i^0(\eta)$ eventually becomes nonempty, and the minimal value of η for which it is nonempty is called the balance number η_0 [1, p. 139]. Thus, by definition,

$$\eta_0 = \min \left\{ \eta : X^0(\eta) = \bigcap_{i=1}^{m} X_i^0(\eta) \neq \emptyset \right\}. \tag{6}$$

The number η_0 defines the minimal equal deviation from (global) optimality for all objective functions.

Generalizing (5), we can allow a different bound η_i for deviation from optimality in each objective function f_i. Then (5) becomes

$$X_i^0(\eta_i) := \left\{ x \in X : f_i(x) - c_i^0 \leq \eta_i,\ \eta_i > 0 \right\}. \tag{7}$$

Again, we are interested in nonempty intersections of sets $X_i^0(\eta_i)$. Galperin gave the following definition [2].

DEFINITION 1. *The point $\eta \in \mathbb{R}^m$ is a balance point if*

$$X_\eta^0 := \bigcap_{i=1}^{m} X_i^0(\eta_i) \neq \emptyset$$

and for every $\eta' \in \mathbb{R}^m$ such that $0 \leq \eta_i' \leq \eta_i$, $i = 1, \dots, m$ and $\eta' \neq \eta$ the set $X_{\eta'}^0 = \emptyset$. The set of all balance points is called the balance set, denoted by Υ.

Several papers have been published on the topic of balance points. In [1], the cubic algorithm has been used to compute the balance number η_0 and the set of η_0-suboptimal solutions of the multiobjective problem (1). Galperin has given some procedures to find balance points in [2]. A relation between the balance space and Pareto optimality has been established in [3]. It has been shown that the balance set is translationally equivalent to the set of efficient solutions of a multicriteria problem. In [4], Pareto analysis and the balance space approach have been compared. The retrieval and use of the balance set has been discussed in [5].

In this paper, we will investigate the so-called apportioned balance numbers [2]. We replace η_i in (7) by $\eta\lambda_i$, where all λ_i, $i = 1, \dots, m$ are nonnegative numbers. Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be fixed. Then we define, similarly to (6) and Definition 1.

DEFINITION 2. *The apportioned balance number $\eta_0(\lambda)$ is the smallest number $\eta \in \mathbb{R}$ such that*

$$X^0_{\eta\lambda} := \bigcap_{i=1}^{m} X^0_i(\eta\lambda_i) = \bigcap_{i=1}^{m} \left\{x : f_i(x) - c^0_i \le \lambda_i\bar{\eta}\right\} \neq \emptyset.$$

The concept of apportioned balance numbers can be very useful in real world applications of multiobjective optimization. Consider a decision maker facing conflicting objectives. He might well have some ideas about how much a solution should deviate from global optimality for each of his criteria. Thus, specifying $\lambda_i = 0$, no deviation at all would be allowed for objective f_i, whereas bigger values of λ_j would indicate that f_j is less important and a bigger deviation is acceptable. An interactive method could then be used to find an initial solution with equal deviation (i.e., the balance number), then allowing the decision maker to specify some weights λ_i to modify the allowable deviation. Continuing this process will finally provide a solution acceptable for the decision maker. Therefore, it is essential to understand apportioned balance numbers, their relations to balance numbers and balance points, and to have methods that allow their fast computation.

In this paper, we will contribute to that. We will provide a min-max formulation, thus opening a way to compute $\eta_0(\lambda)$. The representation of $\eta_o(\lambda)$ in the balance set Υ is discussed, which allows a more efficient way of finding $\eta_0(\lambda)$ for problems where the equation of the balance set is known. The main results of the paper are necessary and sufficient conditions for the property that all balance points can be represented as $\eta_o(\lambda)\lambda$, where $\eta_0(\lambda)$ is an apportioned balance number.

2. A MIN-MAX FORMULATION OF THE APPORTIONED BALANCE NUMBER

In [6], a min-max formulation of the balance number was derived.

THEOREM 1. *The balance number η_0 for the multiobjective optimization problem (1) is determined by the following* min-max *problem:*

$$\eta_0 = \min_{x\in X} \max_{1\le i\le m} \left[f_i(x) - c^0_i\right]. \tag{8}$$

In this section, we generalize the min-max formulation of the balance number, Theorem 1, to apportioned balance numbers.

THEOREM 2. *Let $\lambda_i > 0$ for all $i = 1,\dots,m$. Then the apportioned balance number $\eta_0(\lambda)$ for the multiobjective optimization problem (1) is determined by the following* min-max *problem:*

$$\eta_0(\lambda) = \min_{x\in X} \max_{1\le i\le m} \left[\frac{f_i(x) - c^0_i}{\lambda_i}\right]. \tag{9}$$

PROOF. Note that, due to (2), for $x \in X$ all $f_i(x) \ge c^0_i$, $i = 1,\dots,m$. By Definition 2, we have

$$\begin{aligned}
\eta_0(\lambda) &= \min\left\{\eta : X^0_{\eta\lambda} \neq \emptyset\right\}\\
&= \min\left\{\eta : \exists\, x \in X \text{ such that } f_i(x) - c^0_i \le \lambda_i\eta,\ i = 1,\dots,m\right\}\\
&= \min\left\{\eta : \exists\, x \in X \text{ such that } \frac{f_i(x) - c^0_i}{\lambda_i} \le \eta,\ i = 1,\dots,m\right\}\\
&= \min\left\{\eta : \exists\, x \in X \text{ such that } \max_{1\le i\le m} \frac{f_i(x) - c^0_i}{\lambda_i} \le \eta\right\}.
\end{aligned} \tag{10}$$

Equation (10) represents the following nonlinear optimization problem:

$$\begin{aligned}
&\min \eta\\
&\text{subject to } \max_{1\le i\le m} \left[\frac{f_i(x) - c^0_i}{\lambda_i}\right] \le \eta, \qquad x \in X.
\end{aligned} \tag{11}$$

which, since $f_i(x) \geq c_i^0$ and $\lambda_i > 0$ in (11), has a solution $\eta_0(\lambda) \geq 0$. Thus, minimization with respect to η is implied by minimization with respect to x in (11), yielding

$$\eta_0(\lambda) = \min_{x \in X} \max_{1 \leq i \leq m} \left[\frac{f_i(x) - c_i^0}{\lambda_i} \right]. \tag{12} ∎$$

The condition that $\lambda_i > 0$ is needed in Theorem 2, not only to avoid division by zero, but also because when some $\lambda_i = 0$, then possibly $\eta_0(\lambda)$ need not exist, as we shall see in the next section (cf., Remark 1 and Example 1).

From Theorem 2, we have the possibility to compute apportioned balance numbers as solutions of (nonlinear) min-max problems. For these global optimization problems, one can in general apply, e.g., the cubic algorithm of Galperin [1]. For special problems, specific algorithms are available, e.g., when all f_i are linear and X is a polyhedron then (11) can again be written as a linear programming problem.

3. BALANCE NUMBERS AND BALANCE POINTS—SOME RELATIONS

In this section, we give some relations between (apportioned) balance numbers and balance points. The results show that they are, in a certain sense, represented in the balance set.

PROPOSITION 1. *If $\lambda_i > 0$ for all $i = 1, \ldots, Q$, then the apportioned balance number $\eta_0(\lambda)$ with respect to λ is*

$$\eta_0(\lambda) = \min_{\eta \in \Upsilon} \max_{1 \leq i \leq m} \left\{ \frac{1}{\lambda_i} \eta_i \right\}.$$

PROOF. Let $\hat{\eta} := \min_{\eta \in \Upsilon} \max_{i=1,\ldots,m} \{(1/\lambda_i)\, \eta_i\}$. In our proof, we first show that

$$X_{\hat{\eta}\lambda}^0 = \bigcap_{i=1}^{m} X_i^0(\hat{\eta}\lambda_i) \neq \emptyset,$$

which implies that $\eta_0(\lambda) \leq \hat{\eta}$, due to Definition 2. Second, we show that $\eta_0(\lambda) \geq \hat{\eta}$ by contradiction.

1. To show that $\bigcap_{i=1}^{m} \{x \in X : f_i(x) - c_i^0 \leq \hat{\eta}\lambda_i\} \neq \emptyset$ let $\eta \in \Upsilon$ be such that $\hat{\eta} = \max_{i=1,\ldots,m} \{(1/\lambda_i)\, \eta_i\} = (1/\lambda_j)\, \eta_j$. It follows that

$$\eta_j = \lambda_j \hat{\eta}, \tag{13}$$

$$\eta_i \leq \lambda_i \hat{\eta}, \qquad i = 1, \ldots, m, \quad i \neq j. \tag{14}$$

Because η is a balance point, i.e.,

$$\bigcap_{i=1}^{m} X_i^0(\eta_i) \neq \emptyset,$$

and by (13) and (14), there exists at least one $x \in X$ such that $f_i(x) - c_i^0 \leq \eta_i \leq \lambda_i \hat{\eta}$ $i = 1, \ldots, m$. Therefore,

$$\bigcap_{i=1}^{m} \left\{ x : f_i(x) - c_i^0 \leq \lambda_i \hat{\eta} \right\} \neq \emptyset,$$

and consequently, $\eta_0(\lambda) \leq \hat{\eta}$.

2. To show that also $\eta_0(\lambda) \geq \hat{\eta}$, assume to the contrary that $\eta_0(\lambda) < \hat{\eta}$. Then by Definition 2, there exists at least one $x \in X$ such that $f_i(x) - c_i^0 \leq \lambda_i \eta_0(\lambda)$ $i = 1, \ldots, m$. Now consider the vector

$$f(x) - c^0 := \left(f_1(x) - c_1^0, \ldots, f_m(x) - c_m^0 \right)$$

for such an x. Obviously this implies the existence of a balance point η with the property that

$$\bigcap_{i=1}^{m} \{x \in X : f_i(x) - c_i^0 \le \eta_i\} \ne \emptyset,$$

and additionally,

$$\eta_i \le \lambda_i \eta_0(\lambda) < \lambda_i \hat{\eta},$$

for all $i = 1, \dots, m$, and thus,

$$\max_{i=1,\dots,m} \frac{\eta_i}{\lambda_i} < \hat{\eta}.$$

This contradicts the choice of $\hat{\eta}$. ∎

Recall that, if we choose $\lambda_i = 1$, $i = 1, \dots, m$, then the definition of apportioned balance number becomes the definition of the balance number (6), and from Proposition 1, we immediately obtain the following result.

PROPOSITION 2. *The balance number η_0 is the smallest maximal component of all balance points*

$$\eta_0 = \min_{\eta \in \Upsilon} \max_{i=1,\dots,m} \{\eta_i\}.$$

As in Theorem 2 the condition that $\lambda_i > 0$ in Proposition 1 is necessary, as can be seen in Remark 1 and Example 1, which confirms Remark 1.

REMARK 1.

(i) If $Q \ge 2$ and $\lambda_l = \lambda_k = 0$, $l \ne k$, then $\eta_0(\lambda)$ need not exist.
(ii) If $Q \ge 2$ and $\lambda_j = 0$ and $\lambda_i > 0$ for all $i \ne j$, then even if $\eta_0(\lambda)$ exists $\eta_0(\lambda) = \min_{\eta \in \Upsilon} \max_{i=1,\dots,m}\{(1/\lambda_i)\,\eta_i : \lambda_i > 0\}$ does not necessarily hold.

EXAMPLE 1. We illustrate Remark 1 using an example from [4]. The multiobjective optimization problem (1) is given by $X = [1,2] \subset \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^3$ defined by $f(x) = (x, 2x, -x)$. According to [4], the balance set has the following representation:

$$\Upsilon = \left\{\eta \in \mathbb{R}^3 : \eta_1 \in [0,1],\ \eta_2 = 2\eta_1,\ \eta_3 = 1 - \eta_1\right\}. \tag{15}$$

We select several values for λ.

1. $\lambda = (0.25, 0.25, 0.5)$

   Then a direct calculation as in [4] shows that $\eta_0(\lambda) = 1.6$. To check Proposition 1, we let $\eta_1 = 0.2$. Then $\eta = (0.2, 0.4, 0.8) \in \Upsilon$ and $((1/\lambda_i)\,\eta_i) = (0.8, 1.6, 1.6)$. We also see that

$$\begin{aligned} \eta_1 < 0.2 &\Rightarrow \eta_2 < 0.4, \qquad \eta_3 > 0.8, \\ \eta_1 > 0.2 &\Rightarrow \eta_2 > 0.4, \qquad \eta_3 < 0.8. \end{aligned}$$

   In both cases, $\max_i (1/\lambda_i)\,\eta_i > 1.6$, i.e., Proposition 1 is confirmed in that case.
2. $\lambda = (0, 1, 0)$

   Let us compute the sets $X_i^0(\eta\lambda_i)$. Since $\lambda_1 = \lambda_2 = 0$, for $i = 1$ and $i = 3$, we need to compute $X_i^0(0) = \{x \in X : f_i(x) - c_i^0 \le 0\}$ which equals $\{x : f_i(x) - c_i^0 = 0\}$ by definition (2) of c_i^0. But

$$\begin{aligned} X_1^0(0) &= \left\{x \in X : f_1(x) - c_1^0 = 0\right\} = \{x \in [1,2] : x - 1 = 0\} = \{1\}, \\ X_2^0(0) &= \left\{x \in X : f_3(x) - c_3^0 = 0\right\} = \{x \in [1,2] : -x + 2 = 0\} = \{2\}, \end{aligned} \tag{16}$$

   holds independent of the choice of η. Thus, $\bigcap_{i=1}^m X_i^0(\eta\lambda_i) = \emptyset$ for all $\eta > 0$ and $\eta_0(\lambda)$ does not exist.

3. $\lambda = (0.5, 0, 0.5)$
   We have to find the minimal η such that

$$\{x \in [1,2] : x - 1 \leq 0.5\eta\} \cap \{x \in [1,2] : 2x - 2 = 0\} \cap \{x \in [1,2] : -x + 2 \leq 0.5\eta\} \neq \emptyset. \tag{17}$$

From (16), the second set is $\{1\}$. It is easy to see that for $\eta = 2$ intersection (17) contains only $x = 1$. For $\eta < 2$, the third set implies that $x > 1$, which makes intersection in (17) empty. As a result, we have $\eta_0(\lambda) = 2$. However, $\eta = (0.5, 1, 0.5)$ is a balance point due to (15) with the property

$$\max_{i=1,2,3} \left\{ \frac{1}{\lambda_i} \eta_i : \lambda_i \neq 0 \right\} = \max \left\{ \frac{0.5}{0.5}, \frac{0.5}{0.5} \right\} = 1 < 2.$$

We conclude this section by showing a way to compute the apportioned balance number for given λ for problems where the equation of the balance set is known. Hence, using that equation, a solution satisfying the decision maker's apportioned deviations from global optimality can easily be computed, in case that all $\lambda_i > 0$. Note that in Case (i) of Remark 1, we would require that two of the objectives attain their global minimal values c_i^0 at the same time, which is unlikely to be possible in the case of conflicting objectives.

REMARK 2. The apportioned balance number can be found by solving the following problem on the balance set.

$$\begin{aligned} &\min z \\ &\text{subject to } z \geq \frac{1}{\lambda_i} \eta_i, \qquad i = 1, \dots, m, \\ &\qquad \eta \in \Upsilon. \end{aligned} \tag{18}$$

The result is an immediate consequence of Proposition 1.

EXAMPLE 2. We illustrate the method on the problem of Example 1 and choose again $\lambda = (0.25, 0.25, 0.5)$. In this case, using the balance set equation (15), problem (18) becomes

$$\begin{aligned} &\min z \\ &\text{subject to } z \geq 4\eta_1, \\ &\qquad z \geq 4\eta_2, \\ &\qquad z \geq 2\eta_3, \\ &\qquad \eta_1 \geq 0, \\ &\qquad \eta_1 \leq 1, \\ &\qquad \eta_2 = 2\eta_1, \\ &\qquad \eta_3 = 1 - \eta_1. \end{aligned}$$

This problem is a simple linear programming problem and can easily be solved. The optimal solution is $z^0 = \eta_0(\lambda) = 1.6$.

4. THE EXHAUSTIVE BALANCE SET

In this section, we show that every balance point is equal to an apportioned balance number times the corresponding vector λ of weights for the individual deviations from global optimality. Therefore, by considering all $\lambda \in \mathbb{R}^m$ with $\lambda_i \geq 0$; $i = 1, \dots, m$; $\lambda \neq 0$, and the associated apportioned balance number $\eta_0(\lambda)$, we get a bigger subset Υ_A of the balance space than the balance set. The main results of this section investigate conditions for these two subsets Υ and Υ_A to be equal.

Before we proceed with this analysis, we formally define the set Υ_A. For convenience, we will from now on assume that $\lambda \in \mathbb{R}^m$ is such that $\lambda_i \geq 0$; $i = 1, \ldots, m$ and that $\sum_{i=1}^m \lambda_i = 1$. The latter assumption can be made without loss of generality, since the normalization of λ does not change the relative weights of deviations for the objectives. Let us denote the set of all possible λ vectors (the $m-1$-dimensional simplex) by

$$\Lambda = \left\{ \lambda \in \mathbb{R}^m : \lambda_i \geq 0;\ i = 1, \ldots, m;\ \sum_{i=1}^m \lambda_i = 1 \right\}.$$

Then we can make the following definition.

DEFINITION 3. *The set of apportioned balance vectors is*

$$\Upsilon_A := \{(\eta_0(\lambda)\lambda_1, \ldots, \eta_0(\lambda)\lambda_m) : \lambda \in \Lambda\},$$

where $\eta_0(\lambda)$ is the apportioned balance number for λ, if it exists.

As mentioned above, our first result (Proposition 3) will make use of Definition 3 to show that the balance set is always a subset of Υ_A, yielding another relation between the balance set Υ and apportioned balance numbers, different from the one of Proposition 1.

PROPOSITION 3. *For each $\eta \in \Upsilon$, there exists a $\lambda \in \Lambda$ such that*

$$\eta = (\eta_0(\lambda)\lambda_1, \ldots, \eta_0(\lambda)\lambda_m). \tag{19}$$

In other words: $\Upsilon \subseteq \Upsilon_A$.

PROOF. Let η be a balance number. We have to find an appropriate $\lambda \in \Lambda$ such that (19) is satisfied. Define $\lambda = (\lambda_1, \ldots, \lambda_m)$ by its components

$$\lambda_i := \frac{\eta_i}{\sum\limits_{i \in Q} \eta_i}$$

and let $\hat{\eta} = \sum_{i=1,\ldots,m} \eta_i$. Therefore, we have $\sum_{i=1,\ldots,m} \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, m$. Thus, $\lambda \in \Lambda$. Furthermore, $\eta_i = \lambda_i \hat{\eta}$. It remains to be shown that $\eta_0(\lambda) = \hat{\eta}$. First, note that the sets $X_i^0(\hat{\eta}\lambda_i) = \{x \in X : f_i(x) - c_i^0 \leq \lambda_i \hat{\eta}\}$ are all nonempty, $i = 1, \ldots, m$, because $\lambda_i \hat{\eta} = \eta_i$ and η is a balance point. Therefore, $\eta_0(\lambda) \leq \hat{\eta}$. Now suppose it were true that $\eta_0(\lambda) < \hat{\eta}$. Then by Definition 2, we know that

$$\bigcap_{i=1}^m \{x \in X : f_i(x) - c_i^0 \leq \lambda_i \eta_0(\lambda)\} \neq \emptyset.$$

But $\eta_0(\lambda) < \hat{\eta}$ also implies that $\lambda_i \eta_0(\lambda) \leq \lambda_i \eta_0 = \eta_i$ holds for all $i = 1, \ldots, m$, with strict inequality for at least one index (because at least one $\lambda_i > 0$). These two facts contradict η being a balance point. ∎

It is now a natural question to ask if equality of Υ and Υ_A can be shown. The question is certainly of theoretical interest. But the answer is also relevant for practical reasons. Balance points represent the "best" achievable deviations from optimality in the presence of conflicting objectives. On the other hand, apportioned balance numbers $\eta_0(\lambda)$ and the related vectors of deviation $(\eta_0(\lambda)\lambda_1, \ldots, \eta_0(\lambda)\lambda_m)$ are very useful in practice. Thus, knowing that $\Upsilon = \Upsilon_A$ would imply that by determining the apportioned balance number a "best" possible vector of deviations from individual global minima was achieved. Unfortunately, this equality is not true in general, not even for linear problems, as Example 3 shows.

EXAMPLE 3. We continue with the problem of Examples 1 and 2. If we choose the weights $\lambda = (0.25, 0.25, 0.5)$ again, we know that $\eta_0(\lambda) = 1.6$ (see either Example 2 or [4]). Therefore,

$$\eta_0(\lambda)\lambda = 1.6(0.25, 0.25, 0.5) = (0.4, 0.4, 0.8).$$

But for $\eta_1 = 0.4$, we use the equation of the balance set (15) to calculate the corresponding balance point $\eta = (0.4, 0.8, 0.6)$. Therefore, $\eta_0(\lambda)\lambda$ is no balance point.

Proposition 3 and Example 3 justify the following definition.

DEFINITION 4. *The balance set Υ is called exhaustive if $\Upsilon_A = \Upsilon$.*

Let us now come to the main results of this paper. For the bicriteria case, we give a necessary and sufficient condition for the balance set to be exhaustive. For the general case, $m \geq 3$, a sufficient condition is given. The main ideas behind these results can be explained by an example.

EXAMPLE 4. Consider the following problem:

$$\min_{x \in [0,1]} \left(f_1(x) = 1 - x, f_2(x) = \sqrt{x+1} \right).$$

Some easy computations show that the balance set Υ is given by the expression

$$\Upsilon = \left\{ (\eta_1, \eta_2) : \eta_2 = \sqrt{2 - \eta_1} - 1,\ 0 \leq \eta_1 \leq 1 \right\}.$$

The balance set is exhaustive here, see Figure 1. Observe that the system of equations

$$\eta\lambda_1 = \eta_1,$$
$$\eta\lambda_2 = \eta_2 = \sqrt{2 - \eta_1} - 1,$$

has a unique solution $\eta = \eta_0(\lambda)$ for each choice of $\lambda \in \Lambda$. Note also, that ΥR^2_+ is not convex here, because $f_2(x)$ is a concave function.

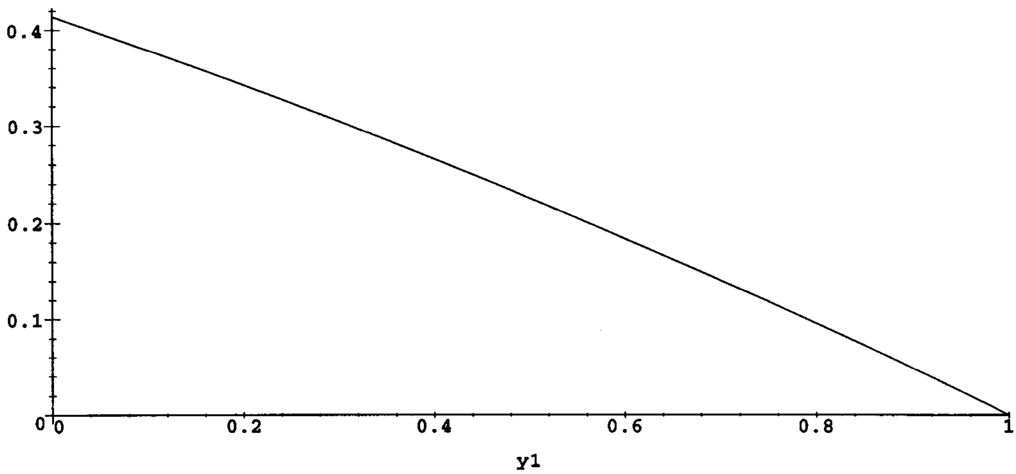

Figure 1. Exhaustive balance set in nonconvex problem.

To check, if $\eta_0(\lambda)\lambda \in \Upsilon$, it is necessary to find the smallest positive number η such that $\eta\lambda \in \Upsilon$, or to confirm that no such η exists. Therefore, we consider the ray $R_\lambda := \{r\lambda : r \in \mathbb{R},\ r \geq 0\}$ and investigate $R_\lambda \cap \Upsilon$. Note that this intersection is either empty or contains exactly one point. Otherwise, there would be $r^1, r^2 \in \mathbb{R}$ such that (wlog) $r^1 < r^2$ and $r^1\lambda \in \Upsilon$, $r^2\lambda \in \Upsilon$. However, because $\lambda \in \Lambda$, this implies that $r^1\lambda_i \leq r^2\lambda_i$, $i = 1, \ldots, m$, and $r^1\lambda \neq r^2\lambda$, which due to the definition of balance points (Definition 1) in turn implies that $r^2\lambda \notin \Upsilon$.

Now assume that $R_\lambda \cap \Upsilon \neq \emptyset$. That the intersection then defines the apportioned balance number $\eta_0(\lambda)$ is shown in Lemma 1.

LEMMA 1. *Let $\lambda \in \Lambda$ and assume that $R_\lambda \cap \Upsilon \neq \emptyset$. Then $\eta_0(\lambda)$ is the unique positive number r such that $r\lambda \in R_\lambda \cap \Upsilon$.*

PROOF. As argued above, there is exactly one $r > 0$ such that $r\lambda \in R_\lambda \cap \Upsilon$. First, we note that, since $r\lambda \in \Upsilon$, i.e., $r\lambda = \eta$ for some $\eta \in \Upsilon$, we know that

$$\bigcap_{i=1}^{m} \left\{ x : f_i(x) - c_i^0 \leq r\lambda_i \right\} \neq \emptyset.$$

Therefore, if $\eta_0(\lambda) \neq r$ by Definition 2, it must be that $\eta_0(\lambda) < r$. Then we would have that

$$\bigcap_{i=1}^{m} \{x : f_i(x) - c_i^0 \leq \eta_0(\lambda)\lambda_i\} \neq \emptyset,$$

with $\eta_0(\lambda)\lambda_i \leq r\lambda_i$, $i = 1, \ldots, m$ and $\eta_0(\lambda)\lambda \neq r\lambda$, contradicting the fact that $r\lambda = \eta \in \Upsilon$. ∎

The ray R_λ is a half-line emanating from the origin, and in Figure 1, we can see that it always intersects Υ, whatever the choice of $\lambda \in \Lambda$. The reason for that seems to be that Υ is connected. That this is indeed true will be shown in Theorem 3. In looking for conditions for Υ to be exhaustive, we first restrict ourselves to the bicriteria case.

We will use the following observation, that immediately follows from Definition 1. Let η^1, $\eta^2 \in \Upsilon$. Then

$$\eta_1^1 < \eta_1^2 \Rightarrow \eta_2^1 > \eta_2^2. \tag{20}$$

We proceed to show that the balance set always intersects the coordinate axes in the balance space $\mathbb{R}_+^2$.

LEMMA 2. *Suppose $m = 2$ and consider the unbalanced multiobjective problem*

$$\min_{x \in X} (f_1(x), f_2(x)),$$

where $X \subset \mathbb{R}^n$ is compact and $f_i :\to \mathbb{R}$ are continuous. Then there exist $\eta^1 = (\eta_1^1, \eta_2^1) \in \Upsilon$ and $\eta^2 = (\eta_1^2, \eta_2^2) \in \Upsilon$ such that $\eta_1^1 = 0$ and $\eta_2^2 = 0$.

PROOF. We prove the existence of η^1 only, the result for η^2 is analogous. According to (3) choose $x^* \in X_1^0$, i.e., $f_1(x^*) = c_1^0$, and in case that X_1^0 is not a singleton such that

$$f_2(x^*) = \min \left\{ f_2(x) : x \in X_1^0 \right\}.$$

(Note that due to compactness of X and continuity of f_1, X_1^0 is compact, too.) Now define

$$\eta^1 := \left(f_1(x^*) - c_1^0, f_2(x^*) - c_2^0\right) = \left(0, f_2(x^*) - c_2^0\right).$$

By the choice of x^*, $x^* \in X_1^0(\eta_1^1) \cap X_2^0(\eta_2^1)$, and also there can be no $\eta \in \mathbb{R}_+^2$ such that $\eta_i^1 \leq \eta_i^2$, $i = 1, 2$, and $\eta \neq \eta^1$ with $X_1^0(\eta_1) \cap X_2^0(\eta_2) \neq \emptyset$. Thus, $\eta^1 \in \Upsilon$. ∎

In particular, Lemma 2 implies that Υ is bounded. We can now prove the main result for exhaustive balance set in bicriteria problems.

THEOREM 3. *Given a bicriteria optimization problem (1) where X is compact and $f : \mathbb{R}^n \to \mathbb{R}^2$ is continuous. Then the balance set Υ is exhaustive if and only if Υ is connected.*

PROOF. In case that problem (1) is balanced, we have $\Upsilon = \{0\}$ and $\eta_0(\lambda) = 0$ for all $\lambda \in \Lambda$ and the result is trivial.

Let us assume that (1) is unbalanced and thus that $\Upsilon \neq \emptyset$. We first note, that due to Lemma 2 for $\lambda = (1, 0)$ or $\lambda = (0, 1)$, the intersection $R_\lambda \cap \Upsilon$ is never empty. Therefore, Lemma 1 implies that $\eta_0(\lambda)\lambda \in \Upsilon$ for these λ, independent of the connectedness assumption. And these intersection points are always on the boundary of Υ. We will therefore, assume that $\lambda_1 > 0$ and $\lambda_2 > 0$ from now on.

1. Assume that Υ is exhaustive, i.e., by Definition 4, for each $\lambda \in \Lambda$, we have $\eta_0(\lambda)\lambda \in \Upsilon$. This implies that for each $\lambda \in \Lambda$, $R_\lambda \cap \Upsilon \neq \emptyset$.

   Suppose Υ is not connected. Then there exist nonempty sets $A_1, A_2 \subset \mathbb{R}^2$ such that $A_1 \cup A_2 = \Upsilon$, $\operatorname{cl} A_1 \cap A_2 = \emptyset$, and $A_1 \cap \operatorname{cl} A_2 = \emptyset$.

   Then $d := \inf\{\|a^1 - a^2\| : a^1 \in \operatorname{cl} A_1,\ a^2 \in \operatorname{cl} A_2\} \geq 0$. Since Υ is bounded, both A_1 and A_2 are bounded as well, and therefore, $\operatorname{cl} A_1$ and $\operatorname{cl} A_2$ are compact. Then let $a^1 \in \operatorname{cl} A_1$

and $a^2 \in \operatorname{cl} A_2$ be such that $d = \|a^1 - a^2\|$ and suppose wlog that $a_1^1 \le a_1^2$, $a_2^1 \ge a_2^2$. Let conv (a^1, a^2) be the line segment defined by a^1, a^2 and choose $y = (y_1, y_2) \in$ int conv (a^1, a^2), or $y = a^1 = a^2$ in case that $\operatorname{conv}(a^1, a^2) = \{a^1\} = \{a^2\}$. Define $\lambda_1 := y_1/(y_1 + y_2)$ and $\lambda_2 := y_2/(y_1 + y_2)$.

We show that for this $\lambda \in \Lambda$, $R_\lambda \cap \Upsilon = \emptyset$, contradicting our assumption. Assume the contrary. Then either $R_\lambda \cap \Upsilon \subset A_1$ or $R_\lambda \cap \Upsilon \subset A_2$. Observe that, by definition, $y \in R_\lambda$. But the choice of y in particular implies that $y \notin \Upsilon$.

Furthermore, for all $\eta \in \Upsilon$ with $\eta_1 < y_1$, we have $\eta_2 > y_2$ and for all $\eta \in \Upsilon$ with $\eta_1 > y_1$, we have $\eta_2 < y_2$, due to the definition of a balance number (see (20)). For $\eta \in \Upsilon$ to be on the ray R_λ, it must hold that $r\lambda = \eta$ for some $r > 0$. Therefore,

$$\begin{aligned} r\lambda_i &= \eta_i, \\ r\,\frac{y_i}{y_1 + y_2} &= \eta_i, \\ r &= \frac{\eta_i}{y_i}\,(y_1 + y_2). \end{aligned}$$

However, the above observation implies that in both cases $r > y_1 + y_2$ and $r < y_1 + y_2$ at the same time. An obvious contradiction.

2. We show that if Υ is not exhaustive, then it is not connected.

   If Υ is not exhaustive, then there is some $\lambda \in \Lambda$ such that $\eta_0(\lambda)\lambda \notin \Upsilon$. Therefore, by Lemma 1, $R_\lambda \cap \Upsilon = \emptyset$. Because $\lambda_i > 0$, $i = 1, 2$, we can describe the ray λ as the half-line

$$R_\lambda = \left\{ (y_1, y_2) : y_2 = \frac{\lambda_2}{\lambda_1}\, y_1; y_1 \ge 0 \right\}.$$

   Now let $A_1 := \{(\eta_1, \eta_2) \in \Upsilon : \eta_2 > \lambda_2/\lambda_1\}$ and $A_2 := \{(\eta_1, \eta_2) \in \Upsilon) : \eta_2 < \lambda_2/\lambda_1\}$. Due to Lemma 2, both A_1 and A_2 are nonempty, because $\eta^1 \in A_1$ and $\eta_2 \in A_2$. As there is no $\eta \in \Upsilon$ such that $\eta_2 = \lambda_2/\lambda_1)\eta_1$, it follows that $\Upsilon = A_1 \cup A_2$, but $\operatorname{cl} A_1 \cap A_2 = \emptyset$ and $A_1 \cap \operatorname{cl} A_2 = \emptyset$, i.e., Υ is not connected. ∎

Sufficient conditions for connectedness can be deduced from conditions for connectedness of the efficient set (or adjoint Pareto set) of multicriteria optimization problem (1) exploiting the fact that $\Upsilon = Y_{\text{eff}} - c^0$, proved in [3]. Such conditions were investigated by [7–11]. Basically, all these conditions assume convexity or quasi-convexity of the objective functions.

However, these conditions are not necessary, as can be seen from Example 4. We give a more general necessary and sufficient condition motivated by an observation of Galperin, that Υ is in general a lower-dimensional surface in the balance space (see [4, p. 534]). In the bicriteria case, where we assume problem (1) to be unbalanced, Υ must therefore be a one-dimensional surface in $\mathbb{R}^2_+$ and can be represented as the graph of a real valued function g.

Define the function $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(r_1) = \begin{cases} r_2, & \text{if } \exists r_2 \in \mathbb{R} : (r_1, r_2) \in \Upsilon, \\ \infty, & \text{otherwise.} \end{cases} \tag{21}$$

By this definition, Υ is the part of the graph of g which is not infinite. Due to observation (20), on the first projection of Υ, $P_1(\Upsilon) = \{\eta_1 : (\eta_1, \eta_2) \in \Upsilon\}$, g will be strictly monotone decreasing. We can now relate connectedness of Υ with continuity of g.

THEOREM 4. *The balance set Υ is connected if and only if g is continuous on the interval $[0, \eta_1^2]$, where $\eta^2 = (\eta_1^2, 0)$ is as in Lemma 2.*

PROOF. Let η^1 and η^2 be the balance points of Lemma 2.

1. The interval $[0, \eta_1^2]$ is obviously connected. If g is continuous on this interval, there is no $r_1 \in [0, \eta_1^2]$ such that $g(r_1) = \infty$ (since $g(\eta_1^2) = 0$ and $g(0) = \eta_2^1 < \infty$). Therefore, the graph of g is connected, i.e., Υ is connected.

2. Suppose that g is not continuous. Two situations may occur.
   - There exists an $r_1 \in [0, \eta_1^2]$ such that $g(r_1) < \infty$ and
   $$a := \lim_{r \to r_1, r > r_1} g(r) < \lim_{r \to r_1, r < r_1} g(r) =: b.$$
   Let $r_2 \in (a, b)$ and define $A_1 := \{\eta \in \Upsilon : \eta_1 \leq r_1\}$ and $A_2 := \{\eta \in \Upsilon : \eta_1 \geq r_1\}$. Then $\Upsilon = A_1 \cup A_2$. Furthermore, (20) implies that if $\eta_1 < r_1$ for some $\eta \in \Upsilon$, then $\eta_2 \geq b$, and if $\eta_1 > r_1$, then $\eta_2 \leq a$. This, and the fact that $a < b$ imply that both $\operatorname{cl} A_1 \cap A_2 = \emptyset$ and $A_1 \cap \operatorname{cl} A_2 = \emptyset$, i.e., Υ is not connected.
   - There exists $r_1 \in [0, \eta_1^2]$ such that $g(r_1) = \infty$. Then there is no $\eta \in \Upsilon$ such that $\eta_1 = r_1$. Then we can define $A_1 := \{\eta \in \Upsilon : \eta_1 < r_1\}$ and $A_2 := \{\eta \in \Upsilon : \eta_1 > r_1\}$. It is obvious that $A_1 \cup A_2 = \Upsilon$, that $\operatorname{cl} A_1 \cap A_2 = A_1 \cap \operatorname{cl} A_2 = \emptyset$, and thus, Υ is not connected. ∎

However, in the general case of more than two criteria, the connectedness assumption is not enough, even imposing convexity in addition does not guarantee exhaustiveness of Υ.

EXAMPLE 5. We consider the convex multicriteria problem with the following convex feasible set and convex objectives.

$$\begin{aligned} X &= [0,1] \times [0,1], \\ f_1(x) &= 1 - x_1, \\ f_2(x) &= x_1 + x_2, \\ f_3(x) &= (x_1 - 0.5)^2 + (x_2 - 0.5)^2. \end{aligned}$$

For $\lambda = (0.25, 0.25, 0.5)$, the apportioned balance number $\eta_o(\lambda)$ is the smallest number η such that

$$\begin{aligned} 1 - x_1 &\leq 0.25\eta, \\ x_1 + x_2 &\leq 0.25\eta, \\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 &\leq 0.5\eta. \end{aligned}$$

These inequalities are satisfied for $\eta = 2$ with $x_1 = 0.5, x_2 = 0$. For $\eta < 2$, they imply $1 - x_1 < 0.5$ and $x_1 + x_2 < 0.5$, which is impossible for $0 \leq x_1, x_2 \leq 1$. Therefore, $\eta_0(\lambda) = 2$. Note also that $c_i^0 = 0$, $i = 1, 2, 3$ and that for $x = (x_1, x_2) = (0.5, 0)$, the objective function values are $(0.5, 0.5, 0.25)$. Thus, $\eta_0(\lambda)\lambda$ is not a balance point.

In Figure 2, the problem is shown, with f_3 expressed in terms of f_1 and f_2.

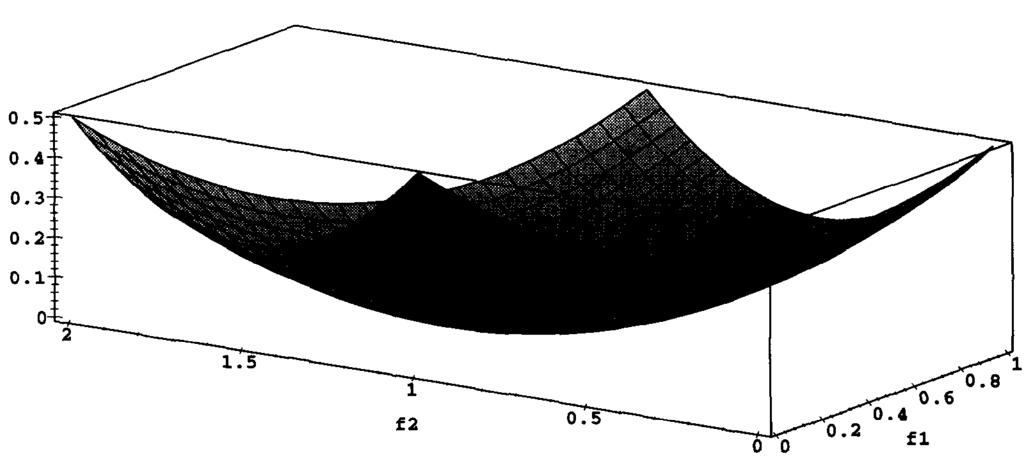

Figure 2. Nonexhaustive balance set in convex problem.

If we make stronger assumptions on the problem, we can prove a sufficient condition for Υ to be exhaustive.

We define the following. Let

$$\hat{c}_i := \max_{\eta \in \Upsilon} \eta_i, \qquad i = 1, \ldots, m.$$

THEOREM 5. *Consider a multicriteria optimization problem. Assume that*

1. $\Upsilon + \mathbb{R}^m_+ := \{\eta + d : \eta \in \Upsilon; d \in \mathbb{R}^m_+\}$ *is convex,*
2. Υ *is closed,*
3. $\bar{c}_i := (0, \ldots, 0, \hat{c}_i, 0, \ldots, 0) \in \Upsilon$ *for all* $i = 1, \ldots, m$.

Then Υ *is exhaustive.*

PROOF. $\Upsilon + \mathbb{R}^m_+$ is convex and closed and vectors $\bar{c}^i$ are contained in Υ for $i = 1, \ldots, m$. Therefore, the convex hull C of the points $\bar{c}^i$ must, by convexity, be contained in $\Upsilon + \mathbb{R}^m_+$. Let $\bar{c} := \sum_{i=1}^m \bar{c}^i$. We note that $(\bar{c} - \mathbb{R}^m_+) \cap (\Upsilon + \mathbb{R}^m_+)$ is bounded and closed, and therefore, compact.

Now let $\lambda \in \Lambda$ and look at the ray R_λ again. We have to show that $R_\lambda \cap \Upsilon \neq \emptyset$ for all $\lambda \in \Lambda$. Lemma 1 shows that $R_\lambda \cap \Upsilon = \{\eta_0(\lambda)\lambda\}$. First of all, it is evident that $R_\lambda \cap C \neq \emptyset$. Therefore, let r_0 be such that

$$r_0\lambda \in C \cap R_\lambda \subseteq \left(\bar{c} - \mathbb{R}^m_+\right) \cap \left(\Upsilon + \mathbb{R}^m_+\right).$$

Because the latter set is compact, the function $d(r) = \|r\lambda\|$ attains its minimum $d(r^*)$ over this set, and furthermore, since $d(r)$ is the distance from the origin to this set along R_λ, $r^*\lambda$ lies on the boundary of $\Upsilon + \mathbb{R}^m_+$. It is also clear from convexity that $r^* \le r_0$.

It remains to be shown that $r^*\lambda \in \Upsilon$. Assume the contrary. Because Υ is closed there must exist some $\epsilon > 0$ such that $(r^*\lambda + \epsilon B) \cap \Upsilon = \emptyset$, where $B = \{y \in \mathbb{R}^m : \|y\| < 1\}$. By the definition of $\Upsilon + \mathbb{R}^m_+$, we can then choose ϵ sufficiently small such that

$$(r^*\lambda + \epsilon B) \cap \left(\Upsilon + \mathbb{R}^m_+\right) = \emptyset.$$

This result contradicts the fact that the point $r^*\lambda$ belongs to the boundary of the (closed) set $(\Upsilon + \mathbb{R}^m_+)$. ∎

Figure 3 provides an illustration of Theorem 5 for the case of $Q = 2$ objective functions.

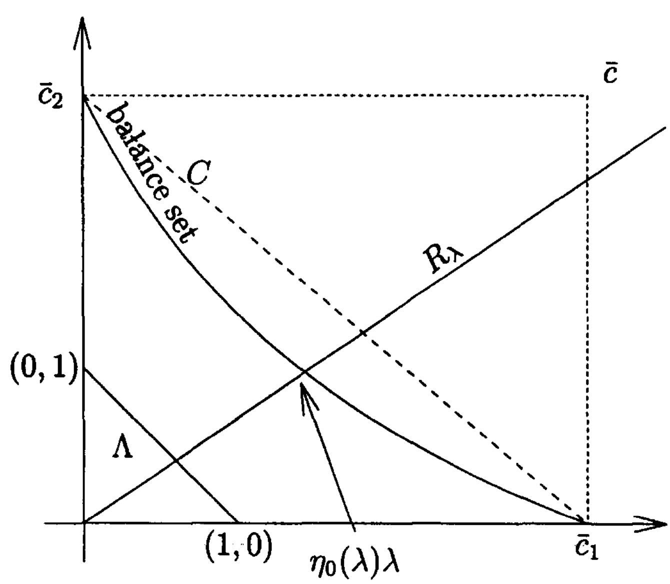

Figure 3. Illustration of the proof of Theorem 5.

We note that the conditions in Theorem 5 ensure that Υ intersects all coordinate axes in the balance space $\mathbb{R}^m_+$ and, in this sense, $\Upsilon + \mathbb{R}^Q_+$ "fills the positive orthant $\mathbb{R}^Q_+$". Let us look at the problem of Example 1 again. Here the first and second assumptions are satisfied because of linearity. However, as can be seen from equation (15), the balance set is a line connecting the points $(0,0,1)$ and $(1,2,0)$ in $\mathbb{R}^3_+$. That Υ is not exhaustive has been shown in Example 3. The same analysis applies to Example 5: Assumptions 1 and 2 are satisfied and still Υ is not exhaustive.

If we relax convexity, it may happen that the balance set is "bent inward" between some of the points $\bar{c}^i$, and therefore, the ray R_λ can pass between Υ and one of the hyperplanes on the boundary of $\mathbb{R}^m_+$ for appropriately chosen λ. Closedness is a technical assumption. Actually, closedness of Υ and the convexity assumption imply that Υ is connected (see the references in the discussion after Theorem 3). Thus, these together are stronger than the condition of Theorem 3.

Consequently, the third condition plays an essential role for the result to be valid. The third condition implies that, also for λ with $\lambda_i = 1$ for some i, R_λ and Υ intersect, namely in $\bar{c}^i$, which is certainly a very strong requirement. However, Example 5 shows that without it, the result is no longer true. Note also that in the bicriteria case, the condition is always satisfied, as shown by Lemma 2.

In the cases where we relax Condition 1 or 3, there may exist an apportioned balance number which does not correspond to a balance point. This happens especially if one of the objectives is redundant, as in Example 2 one of the functions $f_1(x) = x$ and $f_2(x) = 2x$.

Note that the first assumption of Theorem 5 is satisfied if X is a convex set and if all objectives f_i are convex. The first and second are, e.g., satisfied for linear multiple criteria problems if $X = \{x \in \mathbb{R}^n : Ax \leq b,\, x \geq 0\}$ is bounded and if $f(x) = Cx$ is linear.

This discussion indicates that it may be possible to relax the convexity assumption and obtain stronger results also when $m > 2$. This is a topic of future research. For Condition 3 to hold, easily verifiable conditions are unknown as yet, and important results for the application of apportioned balance numbers can be obtained in the future.

5. OUTLINE OF AN INTERACTIVE PROCEDURE

As indicated before, the apportioned balance numbers can be used to derive an interactive procedure for the solution of multicriteria problems (1). The procedure comprises two stages. First, some fundamental analysis about the problem at hand is carried out. The individual minima c_i^0 according to (2) are computed. In case the problem is balanced the decision maker can choose a balanced solution $x^0 \in X^0$, see (4), and the process terminates.

Otherwise, using the information form an η_0-balanced solution, $\eta_0 > 0$ and the minima c_i^0, the decision maker is asked to specify proportions of deviations he prefers. Using the min-max formulation (9), global optimization methods, such as e.g., Galperin's cubic algorithm [1], can be applied to compute the corresponding apportioned balance number and a new solution $x \in X^0_{\eta\lambda}$. This process is continued until the decision maker is satisfied with the current solution. In this process, the results obtained in the previous section can be applied to facilitate the optimization steps (e.g., by using Remark 2), or to obtain further information (e.g., if $\eta_0(\lambda)\lambda$ is a balance point, using the theorems of Section 4).

ALGORITHM 5.1. INTERACTIVE APPORTIONED BALANCE NUMBER PROCEDURE.

Input: Feasible set X and objective functions f of a multicriteria optimization problem (1).
Output: An apportioned balance number $\eta_0(\lambda)$ and a satisfying solution x.
Step 1: Compute the balance number η_0.
Find $x \in X^0$.
If $\eta_0 = 0$ output x and STOP.
Step 2: Present the decision maker with $x, \eta_0, c^0, f(x) - c^0$.

If the decision maker accepts x output x and STOP
else ask the decision maker for weights $\lambda_i \geq 0$; $i = 1, \ldots, m$.

Step 3: Compute $\eta_0(\lambda)$ and $x \in X^0_{\eta\lambda}$.
Goto Step 2.

We illustrate the method using the problem of Example 5.

EXAMPLE 6. As mentioned in Example 5, the individual minima are $c_i^0 = 0$; $i = 1, 2, 3$.

Step 1: The balance number is determined by the smallest number η such that a solution of the system

$$\begin{aligned} 1 - x_1 &\leq \eta, \\ x_1 + x_2 &\leq \eta, \\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 &\leq \eta, \end{aligned}$$

exists. It is easily seen that $\eta_0 = 0.5$, with $x_1 = 0.5, x_2 = 0$ and $X^0(\eta_0) = \{(0.5, 0)\}$. Thus, the problem is unbalanced and we continue with Step 2.

Step 2: The decision maker gets the information that $x = (0.5, 0) \in X^0(\eta_0)$, $\eta_0 = 0.5$, $c^0 = (0, 0, 0), f(x) - c_0 = (0.5, 0.5, 0.25)$. Because he is interested in attaining very good values for f_2, whereas bigger deviations for f_1 and f_3 are acceptable, he specifies $\lambda_1 = 0.5, \lambda_2 = 0, \lambda_3 = 0.5$.

Step 3: To compute $\eta_0(\lambda)$ we find the smallest number η such that the following system has a solution:

$$\begin{aligned} 1 - x_1 &\leq 0.5\eta, \\ x_1 + x_2 &\leq 0, \\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 &\leq 0.5\eta. \end{aligned}$$

The result is $\eta_0(\lambda) = 2$ and $X^0(\eta_0(\lambda)) = \{(0, 0)\}$.

Step 2: With $x = (0, 0)$, $\eta_0(\lambda) = 2$, $f(x) = f(x) - c^0 = (1, 0, 0)$ the decision maker feels that the deviation in f_1 is too big now, and modifies his weights to accept some deviation in f_2. Thus, $\lambda = (0.5, 0.1, 0.4)$.

Step 3: Because the system

$$\begin{aligned} 1 - x_1 &\leq 0.5\eta, \\ x_1 + x_2 &\leq 0.1\eta, \\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 &\leq 0.4\eta, \end{aligned}$$

has no solution for $\eta < 5/3$ we see that $\eta_0(\lambda) = 5/3$, and $X^0(\eta_0(\lambda)) = \{1/6, 0)\}$.

Step 2: The decision maker accepts the solution $x = (1/6, 0)$ with $\eta_0(\lambda) = 5/6$ with $f(x) = (5/6, 1/6, 13/36)$.

Observe that in Example 6, $\eta_0(\lambda)\lambda = (5/6, 1/6, 2/3)$, which is not a balance point as it is different from $f(x) - c^0$. This example shows, that by looking at deviations from global optimality, the decision maker may well be satisfied with a vector of deviations, which does not represent a balance point. However, the final solution chosen in the example yields a balance point $f(x) - c^0 \in \Upsilon$.

6. CONCLUSIONS

In this paper, we have developed some results for the balance space approach to multicriteria optimization. Our focus was on the apportioned balance numbers, which are important for interactive procedures in the balance space context. We derived a min-max formulation, which enables us to compute the apportioned balance number by solving an optimization problem. The main results show that under connectedness conditions the balance set is exhaustive for bicriteria problems. In the general case, we could prove a sufficient condition.

The potential of the concept of apportioned balance numbers in interactive methods has been demonstrated in Section 5. The area offers several possibilities for future research. Possible improvements of the results in Section 4 have already been discussed. But also numerical experiments with an implementation of the interactive algorithm in comparison with older established methods are interesting.

REFERENCES

1. E.A. Galperin, *The Cubic Algorithm for Optimization and Control*, NP Research, Montreal, (1990).
2. E.A. Galperin, Nonscalarized multiobjective global optimization, *Journal of Optimization Theory and Applications* **75** (1), 69–85, (1992).
3. M. Ehrgott, H.W. Hamacher, K. Klamroth, S. Nickel, A. Schöbel and M.M. Wiecek, A note on the equivalence of balance points and Pareto solutions in multiple-objective programming, *Journal of Optimization Theory and Applications* **92** (1), 209–212, (1997).
4. E.A. Galperin, Pareto analysis *vis-a-vis* the balance space approach in multiobjective global optimization, *Journal of Optimization Theory and Applications* **93** (3), 533–545, (1997).
5. E.A. Galperin and M.M. Wiecek, Retrieval and use of the balance set in multiobjective global optimization, *Computers Math. Applic.* **37** (4/5), 111–123, (1999).
6. M. Ehrgott and E.A. Galperin, Min-max formulation of the balance number in multiobjective global optimization, *Computers Math. Applic.*, (this issue).
7. P.H. Naccache, Connectedness of the set of nondominated outcomes in multicriteria optimization, *Journal of Optimization Theory and Applications* **25**, 459–467, (1978).
8. A.R. Warburton, Quasiconcave vector maximization: Connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives, *Journal of Optimization Theory and Applications* **40**, 537–557, (1983).
9. S. Helbig, On the connectedness of the set of weakly efficient points of a vector optimization problem in locally convex spaces, *Journal of Optimization Theory and Applications* **65**, 257–271, (1990).
10. A. Danilidis, N. Hadjisavvas and S. Schaible, Connectedness of the efficient set for three-objective maximization problems, *Journal of Optimization Theory and Applications* **93** (3), 517–524, (1997).
11. M.L. TenHuisen and M.M. Wiecek, On the structure of the non-dominated set for bicriteria programmes, *Journal of Multi-Criteria Decision Analysis* **5**, 232–243, (1996).